On constructions of semigroups

K. P. Shum 1 *

Institute of Mathematics
Yunnan University
Kunming 650091, China
E-mail 1: kpshum@ynu.edu.cn

X. M. Ren 2 † and C. M. Gong 3

Department of Mathematics
Xi’an University of Architecture and Technology
Xi’an 710055, China
E-mail 2: xmren@xauat.edu.cn

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Abstract

The aim of this paper is to present some methods of constructions of semigroups by using the structure theory of semigroups within the class of regular semigroups, the quasi-regular of semigroups and also in the class of abundant semigroups. In particular, some basic notations and structure theorems of some well known semi-groups are exhibited. For example, the Rees matrix semigroups over the 0-group $G^0$ and its generalizations, bands, $E$-ideal quasi-regular semigroups, $C^*$-quasiregular semigroups, $L^*$-inverse semigroups and $Q^*$-inverse semigroups.

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1 Rees matrix semigroups and its generalizations

For notation and terminologies not given in this paper, the reader is referred to [1]-[3], [6], [14]-[15], [17], [24]-[25], [28]-[29], [34] and [39].

We call a semigroup $S$ a completely 0-simple semigroup if $S$ is 0-simple and contains a primitive idempotent.

The following procedures for producing a completely 0-simple semigroups was given by D. Rees in 1940.

We first let $G$ be a group with the identity element $e$ and $I, \Lambda$ non-empty sets. Let $P = (p_{\lambda i})$ be a $\Lambda \times I$ matrix with the entries in an 0-group $G^0(= G \cup \{0\}$). Now, suppose that $P$ is regular in the sense that no row or no column of $P$ consists entirely of zeros.

Formally, we write

\[
(\forall i \in I)(\exists \lambda \in \Lambda) \quad p_{\lambda i} \neq 0,
\]

\[
(\forall \lambda \in I)(\exists i \in I) \quad p_{\lambda i} \neq 0.
\] (1)

Let $S = (I \times G \times \Lambda) \cup \{0\}$, and define a multiplication on $S$ by

\[
(i,a,\lambda) \cdot (b,j,\mu) = \begin{cases} 
(i, ap_{\lambda j}b, \mu) & \text{if } p_{\lambda j} \neq 0, \\
0 & \text{if } p_{\lambda j} = 0,
\end{cases}
\]

\[
(i,a,\lambda) \cdot 0 = 0 \quad (i,a,\lambda) = 0 \cdot 0 = 0. 
\] (2)

The semigroup constructed by this method is denoted by $M^0[G; I, \Lambda; P]$ and will be called the $I \times \Lambda$ matrix semigroup over the 0-group $G^0$ with the regular sandwich matrix $P$. We first state the following well known Rees matrix theorem.

**Theorem 1.1.** (The Rees Theorem, [17]) Let $G^0$ be a 0-group, let $I, \Lambda$ be non-empty sets and $P = (p_{\lambda i})$ a $\Lambda \times I$ matrix with entries in $G^0$. Suppose that $P$ is regular in the sense of (1). Let $S = (I \times G \times \Lambda) \cup \{0\}$, and define a multiplication on $S$ by (2). Then $S$ is a completely 0-simple semigroup.
Conversely, every completely 0-simple semigroup is isomorphic to the semigroup constructed in the above method.

M. V. Lawson in 1990 gave another abstract characterization of the Rees matrix semigroups in [18] as follows:

Let $S$ be a monoid with identity $1$ and zero element $0$, having the group of units $G(S)$. Let $\Lambda$ and $I$ be non-empty sets and $P$ a $\Lambda \times I$ matrix over $S$ with entries $p_{\lambda i}$ where $(\lambda, i) \in \Lambda \times I$. The matrix semigroup $M = M^0(S; I, \Lambda; P)$ is the set triples $I \times S \times \Lambda$ with a zero $0$ adjoined and where we identity all the elements of the form $(i, 0, \lambda)$ with $0$, under a multiplication given by

$$(i, x, \lambda) \cdot (j, y, \mu) = \begin{cases} (i, xp_{\lambda j}y, \mu) & \text{if } p_{\lambda j} \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 1.2.** ([18]) Let $S$ be a Rees semigroup with $e \in U \setminus \{0\}$. Then

(i) $S$ is abundant if and only if $eSe$ is abundant;

(ii) $S$ is regular if and only if $eSe$ is regular;

(iii) $S$ is inverse if and only if $S$ is reduced, $eSe$ is inverse and $Reg_U(S)$ is a subsemigroup (for details, see [18]).

To further generalize the Rees matrix semigroup constructed above, we recently establish in [30] the following construction theorem of semigroups by using semigroupoids.

A semigroupoid is a pair $(S, S^0)$ consisting of a set $S$ of morphisms and a set $S^0$ of objects, together with the functions $\tau : S \to S^0$ and $\omega : S \to S^0$, and a function $\mu$ which is so called “multiplication” from the set $S * S = \{(x, y) \in S \times S \mid \tau(x) = \omega(y)\}$ to $S$; we usually write $xy$ instead of $\mu(x, y)$, and if $(x, y) \in S * S$, then we write $\exists xy$; in addition, the following two axioms hold:

(C1) If $\exists xy$, then $\tau(xy) = \tau(y)$ and $\omega(xy) = \omega(x)$;

(C2) $x(yz) = (xy)z$ whenever the products are defined.

Let $A, B \in S^0$. Then, in this case, the set $\text{Mor}(A, B) = \{x \in S \mid \tau(x) = A \text{ and } \omega(x) = B\}$ is called the Mor-set from $A$ to $B$. A semigroupoid $S$ is said to be strongly connected if each Mor-set $(A, B)$ is non-empty.

Let $I$ and $\Lambda$ be two non-empty sets and $S$ a strongly connected semigroupoid. Define two surjective functions $F : I \to S^0$ and $G : \Lambda \to S^0$.

Now, let $p : \Lambda \times I \to S$ be a function such that

$$p_{(\lambda, i)} \in \text{Mor}(F(i), G(\lambda)).$$

We simply write $p_{(\lambda, i)} = p_{\lambda i}$ so that the entries of the $\Lambda \times I$ matrix $P = (p_{\lambda i})$ are $p_{\lambda i}$.
Let \( M = M(S, F, G; P) \) be the following set
\[
M = \{(i, x, \lambda) \in I \times S \times \Lambda \mid x \in \text{Mor}(G(\lambda), F(i))\},
\]
equipped with the multiplication given by \((i, x, \lambda)(j, y, \mu) = (i, xp_{\lambda}jy, \mu)\).

Then, it is easy to check that the set \( M = \{(i, x, \lambda) \in I \times S \times \Lambda \mid x \in \text{Mor}(G(\lambda), F(i))\} \) forms a semigroup under the above multiplication. We call \( M \) a Rees matrix semigroup over a semigroupoid (for details, see [30]).

## 2 Presentations of bands and its generalizations

A semigroup \( S \) is called a band if every element of \( S \) is an idempotent.

The following general structure theorem for bands was given by M. Petrich in [19].

Let \( X \) be a set. We always write \( T(X)(T^*(X)) \) for the semigroup of all left (right) transformations on the set \( X \). Also, we use the symbol \( \langle \varphi \rangle \) to denote the value of a constant mapping \( \varphi \) acting on the set \( X \).

**Theorem 2.1.** ([17], Theorem 4.4.5) Let \( Y \) be a semilattice and let \( \{E_{\alpha} \mid \alpha \in Y\} \) be a family of pairwise disjoint rectangular bands indexed by \( Y \). For each \( \alpha \), let \( E_{\alpha} = I_{\alpha} \times \Lambda_{\alpha} \), and for each pair \( \alpha, \beta \) of elements of \( Y \) such that \( \alpha \geq \beta \) let \( \Phi_{\alpha, \beta} : E_{\alpha} \to T_{I_{\beta}} \times T_{\Lambda_{\beta}} \) be a morphism, where
\[
a\Phi_{\alpha, \beta} = (\phi_{\alpha, \beta}^{a}, \psi_{\alpha, \beta}^{a}) \quad (a \in E_{\alpha}).
\]
Suppose also that
(i) if \( a = (i, \mu) \in E_{\alpha} \), then \( \phi_{\alpha, a}^{i} \) and \( \psi_{\alpha, a}^{i} \) are constant maps, and
\[
\langle \phi_{\alpha}^{(i, \mu)} \rangle = i, \quad \langle \psi_{\alpha}^{(i, \mu)} \rangle = \mu;
\]
(ii) if \( a \in E_{\alpha}, b \in E_{\beta} \) and \( \alpha \beta = \gamma \), then \( \phi_{\alpha, \beta}^{a} \) and \( \psi_{\alpha, \beta}^{a} \) are constant maps;
(iii) if \( \langle \phi_{\alpha, \beta}^{a} \rangle \) is denoted by \( j \) and \( \langle \psi_{\alpha, \beta}^{a} \rangle \) by \( \nu \), then, for all \( \delta \leq \gamma \),
\[
\phi_{\delta}^{(j, \nu)} = \phi_{\delta}^{a} \phi_{\gamma}^{b}, \quad \psi_{\delta}^{(j, \nu)} = \psi_{\delta}^{a} \psi_{\gamma}^{b}.
\]

Let \( B = \bigcup \{E_{\alpha} \mid \alpha \in Y\} \) and define the product of \( a \) in \( E_{\alpha} \) and \( b \in E_{\beta} \) by
\[
a \ast b = (\langle \phi_{\gamma}^{a} \phi_{\gamma}^{b} \rangle, \langle \psi_{\gamma}^{a} \psi_{\gamma}^{b} \rangle),
\]
where $\gamma = \alpha \beta$. Then $(B, \ast)$ is a band, whose $J$-classes are the rectangular bands $E_\alpha$.

Conversely, every band is determined in this way by a semilattice $Y$, a family of rectangular bands $E_\alpha = I_\alpha \times \Lambda_\alpha$ indexed by $Y$, and a family of morphisms $\Phi_{\alpha,\beta} : E_\alpha \rightarrow T_{I_\beta} \ast T_{\Lambda_\beta}$ ($\alpha, \beta \in Y, \alpha \geq \beta$) satisfying (i), (ii) and (iii).

An element $a$ of a semigroup $S$ is called regular if there exists $x \in S$ such that $a = axa$; An element $a$ of $S$ is called quasi-regular if there exists a natural number $n$ such that $a^n$ is regular. A semigroup $S$ is called regular (quasi-regular), if every element of $S$ is regular (quasi-regular). It is easy to see that quasi-regular semigroups are generalizations of regular semigroups.

As a generalization of bands, X. M. Ren and Y. Q. Guo introduced and investigated the structure of the $E$-ideal quasi-regular semigroups in 1989.

According to [23], we call a semigroup $S$ an $E$-ideal quasi-regular semigroup if $S$ is quasi-regular and $E(S)$ is an ideal of $S$.

For the $E$-ideal quasi-regular semigroups, X. M. Ren and Y. Q. Guo gave the following constructions in [23].

The set $Q$ with a partial operation is called a partial power breaking semigroup if there is a partial binary operation on the set $Q$ such that for any $p, q, r \in Q, (pq)r \in Q$ (well-defined) if and only if $p(qr) \in Q$; in this case $(pq)r = p(qr)$ holds, and for every $a \in Q$, there exists $n \in \mathbb{N}$ such that $a^n \notin Q$.

Let $Y$ be a semilattice and let $\{E_\alpha = I_\alpha \times \Lambda_\alpha \mid \alpha \in Y\}$ be a family of pairwise disjoint rectangular bands. Let $Q$ be a partial power breaking semigroup together with the mapping $\varphi : Q \rightarrow \bigcup_{\alpha \in Y} E_\alpha$ satisfying the following properties:

(i) For any $a, b \in Q$, if $\varphi(a) \in E_\alpha, \varphi(b) \in E_\beta$ and $\alpha \beta = \gamma$ then $ab \in Q$ implies $\varphi(ab) \in E_\gamma$. For every pair $\alpha, \beta \in Y$ with $\alpha \geq \beta$, we can construct two mappings:

$$\Psi_{\alpha,\beta} : \varphi^{-1}(E_\alpha) \rightarrow T_{I_\beta} \ast T_{\Lambda_\beta},$$

$$a \mapsto (\phi^a_\beta, \psi^a_\beta)$$

and

$$\Phi_{\alpha,\beta} : E_\alpha \rightarrow T_{I_\beta} \ast T_{\Lambda_\beta},$$

$$e \mapsto (\phi^e_\beta, \psi^e_\beta)$$

that satisfy the following properties:
(ii) If \( e = (i, j) \in E_\alpha \), then \( \phi^e_\alpha, \psi^e_\alpha \) are constant transformation on \( I_\alpha \) and \( \Lambda_\alpha \), respectively, and \( \langle \phi^e_\alpha \rangle = i, \langle \psi^e_\alpha \rangle = j \). Here, we denote the values of the constant transformations by \( \langle \phi^e_\alpha \rangle \) and \( \langle \psi^e_\alpha \rangle \), respectively.

(iii) 1° If \( e \in E_\alpha \), \( f \in E_\beta \), and \( \delta \leq \gamma = \alpha \beta \), then \( \phi^e_\gamma \phi^f_\gamma \) and \( \psi^e_\gamma \psi^f_\gamma \) are transformations on \( I_\gamma \) and \( \Lambda_\gamma \), respectively. Let \( \langle \phi^e_\gamma \phi^f_\gamma \rangle = i, \langle \psi^e_\gamma \psi^f_\gamma \rangle = j \), we have
\[
\phi^{(i,j)}_\delta = \phi^e_\delta \phi^f_\delta, \psi^{(i,j)}_\delta = \psi^e_\delta \psi^f_\delta.
\]

2° If \( e \in E_\alpha, a \in Q, \varphi(a) \in E_\beta \) and \( \delta \leq \gamma = \alpha \beta \), then \( \phi^a_\beta \phi^b_\gamma, \psi^a_\beta \psi^b_\gamma \) and \( \psi^a_\gamma \psi^b_\gamma \) are constant transformation on \( I_\gamma \) and \( \Lambda_\gamma \), respectively. Let \( \langle \phi^a_\gamma \phi^b_\gamma \rangle = k, \langle \psi^a_\gamma \psi^b_\gamma \rangle = l, \langle \phi^a_\gamma \phi^b_\gamma \rangle = k', \) and \( \langle \psi^a_\gamma \psi^b_\gamma \rangle = l' \), we have
\[
\phi^{(k,l)}_\delta = \phi^a_\delta \phi^b_\delta, \psi^{(k,l)}_\delta = \psi^a_\delta \psi^b_\delta,
\]
\[
\phi^{(k',l')}_\delta = \phi^a_\delta \phi^b_\delta, \phi^{(k',l')}_\delta = \psi^a_\delta \psi^b_\delta.
\]

3° If \( a, b \in Q, ab \notin Q, \varphi(a) \in E_\alpha, \varphi(b) \in E_\beta \) and \( \delta \leq \gamma = \alpha \beta \), then \( \phi^a_\beta \phi^b_\gamma, \psi^a_\beta \psi^b_\gamma \) are constant transformations on \( I_\gamma \) and \( \Lambda_\gamma \), respectively. Let \( \langle \phi^a_\gamma \phi^b_\gamma \rangle = u, \langle \psi^a_\gamma \psi^b_\gamma \rangle = v \), we have
\[
\phi^{(u,v)}_\delta = \phi^a_\delta \phi^b_\delta, \psi^{(u,v)}_\delta = \psi^a_\delta \psi^b_\delta.
\]

(iv) If \( a, b \in Q, ab \in Q, \varphi(a) \in E_\alpha, \varphi(b) \in E_\beta \), and \( \delta \leq \gamma = \alpha \beta \), then
\[
\phi^{ab}_\delta = \phi^a_\delta \phi^b_\delta, \psi^{ab}_\delta = \psi^a_\delta \psi^b_\delta.
\]

We now write \( \sum = Q \bigcup_{a \in \gamma} E_\alpha \) and define an operation \( * \) on \( \sum \) as follows:

a) If \( a, b \in Q \) and \( ab \in Q \), then \( a * b = ab \);

If \( a, b \in Q, \varphi(a) \in E_\alpha, \varphi(b) \in E_\beta \) and \( \alpha \beta = \gamma \), but \( ab \notin Q \), then
\[
a * b = (\langle \phi^a_\gamma \phi^b_\gamma \rangle, \langle \psi^a_\gamma \psi^b_\gamma \rangle).
\]

b) If \( e \in E_\alpha, a \in Q, \varphi(a) \in E_\beta \), and \( \alpha \beta = \gamma \), then
\[
a * e = (\langle \phi^a_\gamma \phi^b_\gamma \rangle, \langle \psi^a_\gamma \psi^b_\gamma \rangle),
\]
\[
e * a = (\langle \phi^a_\gamma \phi^b_\gamma \rangle, \langle \psi^a_\gamma \psi^b_\gamma \rangle).
\]
c) If $e \in E_\alpha$, $f \in E_\beta$, and $\alpha \beta = \gamma$, then

$$e * f = (\langle \phi_{\gamma}^e \phi_{\gamma}^f \rangle, \langle \psi_{\gamma}^e \psi_{\gamma}^f \rangle).$$

The above system consisting of $\sum$ and the operation $*$ on $\sum$ is now denoted by $\sum = \sum(Q, \bigcup_{\alpha \in Y} E_\alpha, \Psi, \Phi, \varphi)$.

It is easy to show that $\sum = \sum(Q, \bigcup_{\alpha \in Y} E_\alpha, \Psi, \Phi, \varphi)$ is a semigroup, that is, the above operation $*$ on $\sum$ is associative. We give the following picture to illustrate the relationship between the construction mappings above, where $Q_\alpha = \varphi^{-1}(E_\alpha)$.

Theorem 2.2. ([23]) Let $S$ be a semigroup. Then $S$ is an $E$-ideal quasi-regular semigroup if and only if $S$ is isomorphic to some semigroup of type $\sum = \sum(Q, \bigcup_{\alpha \in Y} E_\alpha, \Psi, \Phi, \varphi)$.

3 \(\Delta\)-products and generalized \(\Delta\)-products

A regular semigroup $S$ is called a left $C$-semigroup (in brevity, LC-semigroup) if for any $a \in S$, $aS \subseteq Sa$. In 1991, Zhu, Guo and Shum gave the following characterization theorem for the left $C$-semigroups in [40].

Theorem 3.1. ([40]) Suppose that $S$ is an orthodox semigroup with a band $E$ of idempotents. Then the following statements on $S$ are equivalent:

(i) $S$ is a left $C$-semigroup;
(ii) $(\forall e \in E) eS \subseteq Se$;
(iii) $(\forall e \in E)(\forall a \in S) eae = ea$;
(iv) $D^S \cap (E \times E) = L^E$;
(v) $S$ is a semilattice of left groups;
(vi) $L = J$ is a semilattice congruence on $S$. 

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In studying the structure theory of left \( C \)-semigroups, Guo, Ren and Shum [12] introduced the concept of \( \Delta \)-products of semigroups as follows:

Let \( Y \) be a semilattice. Let \( T = \bigcup_{\alpha \in Y} T_\alpha \) be a semilattice of semigroups \( T_\alpha \) and \( I = \bigcup_{\alpha \in Y} I_\alpha \) a semilattice partition of the set \( I \) on the semilattice \( Y \). For each \( \alpha \in Y \), write \( S_\alpha = T_\alpha \times I_\alpha \); For any \( \alpha, \beta \in Y, \alpha \geq \beta \), define the following mapping

\[
\Psi_{\alpha,\beta} : S_\alpha \rightarrow T_\beta,
\]

\[
a \mapsto \psi_{\alpha,\beta}^a,
\]

satisfying the following conditions:

(P1) If \((u, i) \in S_\alpha, i' \in I_\alpha\), then \(\psi_{\alpha, \alpha}^{(u, i)} i' = i\);

(P2) If \((u, i) \in S_\alpha, (v, j) \in S_\beta\), then

(a) \(\psi_{\alpha, \alpha \beta}^{(u, i)} \psi_{\beta, \alpha \beta}^{(v, j)}\) are constant values mappings on \(I_{\alpha \beta}\), denote the value by \(\langle \psi_{\alpha, \alpha \beta}^{(u, i)} \psi_{\beta, \alpha \beta}^{(v, j)} \rangle\);

(b) If \(\alpha \beta \geq \delta, \langle \psi_{\alpha, \alpha \beta}^{(u, i)} \psi_{\beta, \alpha \beta}^{(v, j)} \rangle = k\), we have \(\psi_{\alpha \beta, \delta}^{(u, i, k)} = \psi_{\alpha, \delta}^{(u, i)} \psi_{\beta, \delta}^{(v, j)}\).

Define a multiplication on the set \( S \) by

\[
(u, i) \ast (v, j) = (uv, \langle \psi_{\alpha, \alpha \beta}^{(u, i)} \psi_{\beta, \alpha \beta}^{(v, j)} \rangle), \quad (u, i) \in S_\alpha, (v, j) \in S_\beta.
\]

where \(uv\) is the product of \( u \) and \( v \) in the semigroup \( T \).

It is easy to verify that \( S = \bigcup_{\alpha \in Y} S_\alpha \) with the operation given above forms a semigroup. The semigroup \( S \) constructed above is called a \( \Delta \)-product of a semigroup \( S \) and a set \( I \) with respect to a semilattice \( Y \) and a structure map \( \psi \) is denoted by \( S = T \Delta Y, \psi I \).

The \( \Delta \)-product of a semigroup \( S = \bigcup_{\alpha \in Y} (T_\alpha \times I_\alpha) \) with a set \( I = \bigcup_{\alpha \in Y} I_\alpha \) with respect to a semilattice \( Y \) and a structure map \( \psi \) can be expressed by the following diagram, where \( \alpha \geq \beta \) for all \( \alpha, \beta \in Y \).
**Theorem 3.2.** ([12]) Let $T = [Y; G_{\alpha}, \varphi_{\alpha, \beta}]$ be a strong semilattice of group $G_{\alpha}$ and let $I = \bigcup_{\alpha \in Y} I_{\alpha}$ be a semilattice decompositions of a left regular band $I$ for left zero bans $I_{\alpha}$. Then the $\Delta$-product $S = T \Delta_{Y, \Psi} I$ of $T$ and $I$ with respect to $Y$ is a LC-semigroup; conversely, every LC-semigroup $S$ can be constructed in the above fashion.

According to [36], a quasi-regular semigroup $S$ is called a $C^*$-quasiregular semigroup if for any $e \in E(S)$, the mapping $\psi_e : S^1 \to eS^1e$ defined by $x \mapsto exe$ is a semigroup homomorphism and $RegS$ is an ideal of $S$.

There were some characterization theorems of $C^*$-quasiregular semigroups given by Shum, Ren and Guo in [36].

**Theorem 3.3.** ([36]) The following statements are equivalent for a semigroup $S$:

(i) $S$ is a $C^*$-quasiregular semigroup;
(ii) $S$ is a quasi-completely regular semigroup in which $RegS$ is an ideal of $S$ and $E(S)$ is a regular band;
(iii) $S$ is a quasi-completely regular semigroup such that $eS \cup Se \subseteq RegS$ and the mapping $\varphi_e : E(S) \to eE(S)e$ defined by $f \mapsto efe$ is a semigroup homomorphism for all $e \in E(S)$;
(iv) $S$ is a semilattice of quasi-rectangular groups such that

$$(\forall a \in S)(\exists m \in N) \ a^mS \cup Sa^m \subseteq RegS$$

and $E(S)$ is a regular band;
(v) $S$ is a nil-extension of a quasi-$C$-semigroup.

It is well-known that the structure of completely regular semigroups can be described by the translational hull of semigroups (see M. Petrich in [20]). Inspired by the idea of M. Petrich, we can also construct a quasi-completely regular semigroup $S$ by using the translations on the semigroup $S$.

To obtain the structure of $C^*$-quasiregular semigroups, we consider a more general constructions method for semigroups rather than the $\Delta$-product structure. We call this new structure the generalized $\Delta$-product structure.

We first cite the following concepts.

A mapping $\theta$ from a power breaking partial semigroup $Q$ to another one is called a partial homomorphism if $(ab)\theta = a\theta b\theta$, whenever $a, b, ab \in Q$.  

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We are now ready to state the definition of generalized $\Delta$-product of semigroups.

(I) Let $\tau$ be a partial homomorphism from a power breaking partial semigroup $Q$ to a semilattice $Y$; Then, we write $Q_\alpha = \tau^{-1}(\alpha)$, for any $\alpha \in Y$.

(II) Let $T = [Y, T_\alpha, \xi_{\alpha\beta}]$ be a strong semilattice of semigroups $T_\alpha$, where $\xi_{\alpha\beta}$ is the structure homomorphism. Let $I = \bigcup_{\alpha \in Y} I_\alpha$ and $\Lambda = \bigcup_{\alpha \in Y} \Lambda_\alpha$ be a semilattice partition for the set $I$ and for the set $\Lambda$ on the semilattice $Y$ respectively. It is well-known that if $T_\alpha$ are groups then the strong semilattice $T = [Y, T_\alpha, \xi_{\alpha\beta}]$ is a Clifford semigroup.

For every $\alpha \in Y$, form the following three sets, namely, the sets

\[ \begin{align*}
S^0_\alpha &= Q_\alpha \cup T_\alpha, \\
S^\ell_\alpha &= Q_\alpha \cup (I_\alpha \times T_\alpha), \\
S^r_\alpha &= Q_\alpha \cup (T_\alpha \times \Lambda_\alpha).
\end{align*} \]

(III) For any $\alpha, \beta \in Y$ with $\alpha \geq \beta$, define the following mapping

\[ \theta_{\alpha,\beta} : S^0_\alpha \rightarrow T_\beta \text{ by } a \mapsto a\theta_{\alpha,\beta}, \]

and we require that $\theta_{\alpha,\beta}$ satisfies the following conditions.

(P1) (i) $\theta_{\alpha,\beta}|_{T_\alpha} = \xi_{\alpha\beta};$
(ii) If $a \in Q_\alpha$ and $\alpha \geq \beta \geq \gamma$, then $a\theta_{\alpha,\beta}\theta_{\beta\gamma} = a\theta_{\alpha,\gamma};$
(iii) If $a \in Q_\alpha, b \in Q_\beta$ and $ab \in Q_{\alpha\beta}$ with $\alpha \beta \geq \delta$, then

\[ (ab)\theta_{\alpha,\beta,\delta} = a\theta_{\alpha,\delta}b\theta_{\beta,\delta}. \]

(IV) For $\alpha, \beta \in Y$ with $\alpha \geq \beta$, define the following two mappings $\varphi_{\alpha,\beta}$ and $\psi_{\alpha,\beta}$

\[ \begin{align*}
\varphi_{\alpha,\beta} : S^\ell_\alpha &\rightarrow T(I_\beta) \text{ by } a \mapsto \varphi^a_{\alpha,\beta}; \\
\psi_{\alpha,\beta} : S^r_\alpha &\rightarrow T^*(\Lambda_\beta) \text{ by } a \mapsto \psi^a_{\alpha,\beta}.
\end{align*} \]

Let $\varphi_{\alpha,\beta}$ and $\psi_{\alpha,\beta}$ satisfy the following conditions (P1), (p2), (P2)* and (P3)* respectively.

(P2) If $(i, g) \in I_\alpha \times T_\alpha$ and $j \in I_\alpha$, then $\varphi^{(i,g)}_{\alpha,a} j = i$;
(P2)* If $(g, \lambda) \in T_\alpha \times \Lambda_\alpha$ and $\mu \in \Lambda_\alpha$, then $\mu\psi^a_{\alpha,g} = \lambda$;

For the sake of convenience, we write $(i, g)\theta_{\alpha,\beta} = g\theta_{\alpha,\beta}$ and $(g, \lambda)\theta_{\alpha,\beta} = g\theta_{\alpha,\beta}$ for any $(i, g) \in I_\alpha \times T_\alpha$ and $(g, \lambda) \in T_\alpha \times \Lambda_\alpha$.

(P3) Let $\alpha, \beta$ and $\delta \in Y$ with $\alpha \beta \geq \delta$.
(i) If $a \in S^\ell_\alpha, b \in S^\ell_\beta$ and $ab \in Q_{\alpha\beta}$, then $\varphi^a_{\alpha,\beta}\delta = \varphi^a_{\alpha,\delta}\psi^b_{\beta,\delta}$;
(ii) If $a \in S^\ell_\alpha, b \in S^\ell_\beta$ and $ab \notin Q_{\alpha\beta}$, then $\varphi^a_{\alpha,\beta}\psi^b_{\beta,\alpha}$ is a constant mapping acting on the set $I_{\alpha\beta}$.
Let \( k = \langle \varphi_{a,\alpha,\beta} \varphi_{b,\alpha,\beta} \rangle \) be the constant value of \( \varphi_{a,\alpha,\beta} \varphi_{b,\alpha,\beta} \) and \( g = a\theta_{a,\alpha,\beta} b\theta_{\beta,\alpha,\beta} \). Then
\[
\varphi_{a,\alpha,\delta}^{(k,g)} = \varphi_{a,\delta}^{a} \varphi_{b,\delta}^{b}.
\]

(P3*) Let \( \alpha, \beta \) and \( \delta \) \( Y \) with \( \alpha \beta \geq \delta \).

(i) If \( a \in S^\alpha_a, b \in S^\beta_b \) and \( ab \in Q_{\alpha\beta} \) then \( \psi_{a,\alpha,\delta}^{ab} = \psi_{a,\delta}^{a} \psi_{b,\delta}^{b} \).

(ii) If \( a \in S^\alpha_a, b \in S^\beta_b \) and \( ab \notin Q_{\alpha\beta} \), then \( \psi_{a,\alpha,\beta}^{a} \psi_{b,\alpha,\beta}^{b} \) is a constant mapping acting on the set \( \Lambda_{\alpha\beta} \).

Let \( u = \langle \psi_{a,\alpha,\beta}^{a} \psi_{b,\alpha,\beta}^{b} \rangle \) be the constant value of \( \psi_{a,\alpha,\beta}^{a} \psi_{b,\alpha,\beta}^{b} \) and \( \nu = a\theta_{a,\alpha,\beta} b\theta_{\beta,\alpha,\beta} \). Then
\[
\psi_{a,\alpha,\delta}^{(u,\nu)} = \psi_{a,\delta}^{a} \psi_{b,\delta}^{b}.
\]

(V) Now, form the set \( S = \bigcup_{\alpha \in Y} S_{\alpha} = \bigcup_{\alpha \in Y} (Q_{\alpha} \cup (I_{a} \times T_{\alpha} \times \Lambda_{\alpha})) \) and define a binary operation \( "*" \) on \( S \) satisfying the following conditions:

[M1] If \( a \in Q_{\alpha}, b \in Q_{\beta} \) and \( ab \in Q_{\alpha\beta} \), then \( a \ast b = ab \).

[M2] If \( a \in Q_{\alpha}, b \in Q_{\beta} \) and \( ab \notin Q_{\alpha\beta} \), then
\[
(a, g, \lambda) \ast a = (\langle \varphi_{a,\alpha,\beta}^{(a)} \varphi_{b,\alpha,\beta}^{(g)} \rangle, a\theta_{a,\alpha,\beta} b\theta_{\beta,\alpha,\beta}, \langle \psi_{a,\alpha,\beta}^{a} \psi_{b,\alpha,\beta}^{b} \rangle).
\]

[M3] If \( a \in Q_{\alpha}, (i, g, \lambda) \in I_{\beta} \times T_{\beta} \times \Lambda_{\beta} \) then
\[
a \ast (i, g, \lambda) = (\langle \varphi_{a,\alpha,\beta}^{a} \varphi_{b,\alpha,\beta}^{(i)} \rangle, a\theta_{a,\alpha,\beta} g\theta_{\beta,\alpha,\beta}, \langle \psi_{a,\alpha,\beta}^{a} \psi_{b,\alpha,\beta}^{(i,\lambda)} \rangle).
\]

[M4] If \( (i, g, \lambda) \in I_{a} \times T_{\alpha} \times \Lambda_{\alpha}, (j, h, \mu) \in I_{\beta} \times T_{\beta} \times \Lambda_{\beta} \) then
\[
(i, g, \lambda) \ast (j, h, \mu) = (\langle \varphi_{a,\alpha,\beta}^{a} \varphi_{b,\alpha,\beta}^{(i)} \rangle, g\theta_{a,\alpha,\beta} h\theta_{\beta,\alpha,\beta}, \langle \psi_{a,\alpha,\beta}^{a} \psi_{b,\alpha,\beta}^{(i,\lambda, j, h, \mu)} \rangle).
\]

It can be verified, by routine checking, that \( (S, \ast) \) is a semigroup.

Now we write \( \Sigma = \{ \varphi_{a,\alpha,\beta}, \psi_{a,\alpha,\beta}, \theta_{\alpha,\beta} \mid \alpha, \beta \in Y, \alpha \geq \beta \} \) and call it the structure mapping of the semigroup \( S = \bigcup_{\alpha \in Y} (Q_{\alpha} \cup (I_{a} \times T_{\alpha} \times \Lambda_{\alpha})) \).

We can use the following picture to illustrate the relationship of the structure mappings above, where \( \alpha \geq \beta \) for all \( \alpha, \beta \in Y \).
Summarizing all the above steps, we formulate the following definition.

**Definition 3.4.** ([36]) The above constructed semigroup $S$ is called the generalized $\Delta$-product of the power breaking partial semigroup $Q$, the semigroup $T$, the sets $I$ and $\Lambda$ with respect to the semilattice $Y$ and the structure mapping $\Sigma$. Denote this semigroup by $S = \Delta_{Y,\Sigma}(Q, I, T, \Lambda)$.

Now we state a construction theorem for a $C^*$-quasiregular semigroup.

**Theorem 3.5.** ([36]) Let $Y$ be a semilattice, $Q$ be a power breaking partial semigroup, $G = [Y, G_\alpha, \xi_{\alpha,\beta}]$ be a strong semilattice of groups $G_\alpha$, $I = \bigcup_{\alpha \in Y} I_\alpha$ and $\Lambda = \bigcup_{\alpha \in Y} \Lambda_\alpha$ be a left regular band and a right regular band, respectively. Then, the generalized $\Delta$-product $\Delta_{Y,\Sigma}(Q, I, G, \Lambda)$ is a $C^*$-quasiregular semigroup.

Conversely, every $C^*$-quasiregular semigroup can be constructed by a generalized $\Delta$-product $\Delta_{Y,\Sigma}(Q, I, G, \Lambda)$.

**A constructed example**

We here construct an example of a non-trivial $C^*$-quasiregular semigroup to illustrate Theorem 3.5.

**Step I** Let $Y = \{\alpha, \beta, \alpha\beta\}$ be a basic semilattice.

**Step II** Let $T_\alpha = \{e_0\}, T_\beta = \{g_0\}$ and $T_{\alpha\beta} = \{w_0, a_0, b_0\}$ be groups. Mount each of these groups on its corresponding vertex of $Y$. Thus $T = [Y; T_\alpha; \xi_{\alpha,\beta}]$ is a strong semilattice of groups which is known as the Clifford semigroup.

The Clifford semigroup $T = [Y; T_\alpha; \xi_{\alpha,\beta}]$ is displayed by the following diagram.
Step III Let $I_\alpha = \{i,j\}$, $I_\beta = \{k,l\}$ and $I_{\alpha\beta} = \{m\}$ be left zero bands. Form a set $I = \bigcup_{\alpha \in Y} I_\alpha$. Similarly, let $\Lambda_\alpha = \{i',j'\}$, $\Lambda_\beta = \{k',l'\}$ and $\Lambda_{\alpha\beta} = \{m'\}$ that are right zero bands. Form a set $\Lambda = \bigcup_{\alpha \in Y} \Lambda_\alpha$.

Step IV On each vertex of the semilattice $Y$, we form the Cartesian product of the left zero bands and the groups, namely $I_\alpha \times T_\alpha$, $\alpha \in Y$. Similarly, we form $T_\alpha \times \Lambda_\alpha$, $\alpha \in Y$. By combining, we have $S_\alpha^{(1)} = I_\alpha \times T_\alpha \times \Lambda_\alpha$, $\alpha \in Y$. Let $(i,e_0,i') = e,(i,e_0,j') = f,(j,e_0,i') = e',(j,e_0,j') = f';(k,g_0,k') = g,(k,g_0,l') = h,(l,g_0,k') = g',(l,g_0,l') = h';(m,w_0,m') = w,(m,a_0,m') = u$ and $(m,b_0,m') = v$ respectively. Mounting the above semigroups on the corresponding vertices on the semilattice $Y$, we obtain the following diagram.

Step V Let $Q = Q_\alpha \cup Q_\beta \cup Q_{\alpha\beta}$, where $Q_\alpha = \{a\}$, $Q_\beta = \{b\}$ and $Q_{\alpha\beta} = \emptyset$. Suppose that $a^2, b^2, ab$ and $ba$ are not in $Q$. Then $Q$ is a power breaking partial semigroup. The presence of such a $Q$ in a semigroup $S$ is an important ingredient to make the semigroup $S$ to be a quasiregular semigroup. In view of this fact, we form the sets $S_\alpha^o = Q_\alpha \cup T_\alpha = \{a,e_0\}$, $S_\beta^o = Q_\beta \cup T_\beta = \{b,g_0\}$ and $S_{\alpha\beta}^o = \{w_0,a_0,b_0\}$ as the components of $S$.

Construct the structure maps of $S$ by $\theta_{\gamma,\delta} : S_\gamma^o \to T_\delta$ for $\gamma \geq \delta$. We obtain the following diagram.
Step VI For every $\alpha \in Y$, $S^l_\alpha = Q_\alpha \cup (I_\alpha \times T_\alpha)$ and $S^r_\alpha = Q_\alpha \cup (T_\alpha \times \Lambda_\alpha)$. Construct the structure mappings $\varphi_{\gamma,\delta} : S^l_\gamma \rightarrow T(I_\delta)$ and $\psi_{\gamma,\delta} : S^r_\gamma \rightarrow T^*(\Lambda_\delta)$ for $\gamma \geq \delta$ on $Y$ as shown below:

\[
\begin{align*}
\varphi_{\alpha,\alpha} &: (i, e_0) \rightarrow \begin{pmatrix} i & j \\ i & i \end{pmatrix}; \\
& (j, e_0) \rightarrow \begin{pmatrix} i & j \\ j & j \end{pmatrix}; \\
\psi_{\alpha,\alpha} &: (e_0, i') \rightarrow \begin{pmatrix} i' & j' \\ i' & i' \end{pmatrix}; \\
& (e_0, j') \rightarrow \begin{pmatrix} i' & j' \\ j' & j' \end{pmatrix};
\end{align*}
\]

Clearly, the mapping $\varphi_{\alpha,\alpha}$ and $\varphi_{\beta,\alpha\beta}$ are the trivial mappings which map respectively $S^l_\alpha$ and $S^l_\beta$ onto $T(I_{\alpha\beta})$. Dually, the mappings $\psi_{\alpha,\alpha}$ and $\psi_{\beta,\alpha\beta}$ can be similarly defined.

Step VII Link up all the above semigroup components $S_\alpha = Q_\alpha \cup (I_\alpha \times T_\alpha \times \Lambda_\alpha) = Q_\alpha \cup S^{(1)}_\alpha$ and the corresponding transformation semigroups $T(I_\alpha)(T^*(\Lambda_\alpha))$ for any $\alpha \in Y$ on the vertices of the semilattice $Y$ via the structure mapping. Thus the multiplication “*” on $S$ can be defined accordingly, by considering all the structure mappings on the
components of the semigroup \( S = \bigcup_{\alpha \in Y} (Q_{\alpha} \cup (I_{\alpha} \times T_{\alpha} \times \Lambda_{\alpha})) \).

The following diagram displays the generalized \( \Delta \)-product structure of the constructed example of the \( C^* \)-quasiregular semigroup.

**Step VIII** Summing up all the above steps, the multiplication “\( * \)” of the semigroup \( S = \bigcup_{\alpha \in Y} S_{\alpha} = \bigcup_{\alpha \in Y} (Q_{\alpha} \cup (I_{\alpha} \times Q_{\alpha} \times T_{\alpha})) \) is defined.

The Cayley table of the semigroup \( S = \{ a, b, e, f, e', f', g, h, g', h', w, u, v \} \) is shown below:

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15
4 Left wreath products and wreath products

The concept of an abundant semigroup was first introduced by J. B. Fountain in 1971 [5]. To state the definition of an abundant semigroup, we first cite a set of relations on the semigroup \( S \) called the Green “\(*\)” relations on a semigroup \( S \).

\[
L^* = \{(a, b) \in S \times S \mid (\forall x, y \in S^1) ax = ay \Leftrightarrow bx = by\},
\]
\[
R^* = \{(a, b) \in S \times S \mid (\forall x, y \in S^1) xa = ya \Leftrightarrow xa = yb\},
\]
\[
H^* = L^* \land \ R^*,
\]
\[
D^* = L^* \lor \ R^*,
\]
\[
J^* = \{(a, b) \in S \times S \mid J^*(a) = J^*(b)\},
\]

where \( J^*(a) \) denote the principal \(*\)-ideal generated by the element \( a \) in \( S \) (see [5]).

Clearly, on any semigroup \( S \) we have \( L \subseteq L^* \) and \( R \subseteq R^* \). It is easy to see that for regular elements \( a, b \in S \), \( (a, b) \in L^* \) if and only if \( (a, b) \in L \). Moreover, we can easily see that \( L^* \) is a right congruence and \( R^* \) is a left congruence on \( S \), respectively.

An abundant semigroup \( S \) is a semigroup in which each \( L^* \)-class and each \( R^* \)-class contains an idempotent. It is clear that a regular semigroup is abundant. In fact, abundant semigroups can be regarded as natural generalizations of regular semigroups.

An abundant semigroup \( S \) is called an \( L^* \)-inverse semigroup if \( S \) is an IC semigroup whose idempotents form a left regular band (for details, see [26]).

To obtain structure of \( L^* \)-inverse semigroups, the concept of left wreath product of semigroups was introduced by Ren and Shum in [26].

Let \( \Gamma \) be a type \( A \) semigroup with semilattice \( Y \) of idempotents. Let \( B = \bigcup_{\alpha \in Y} B_\alpha \) be a semilattice decomposition of a left regular band \( B \) into left zero bands \( B_\alpha \).

Because the type \( A \) semigroup \( \Gamma \) is abundant, we can always identify the element \( \gamma \in \Gamma \) by its corresponding idempotent \( \gamma^\dagger \in R^*_\gamma(\Gamma) \cap E \) or by \( \gamma^* \in L^*_\gamma(\Gamma) \cap E \), respectively. Moreover, since the type \( A \) semigroup \( \Gamma \) is also an IC abundant semigroup, there is a connecting isomorphism \( \eta : \langle \omega^\dagger \rangle \rightarrow \langle \omega^* \rangle \) such that \( \alpha \omega = \omega(\alpha \eta) \) for any \( \alpha \in \langle \omega^\dagger \rangle \) and \( \omega \in \Gamma \).

Now, we form the set \( B \bowtie \Gamma = \{(e, \gamma) \mid e \in B_{\gamma^\dagger}, \gamma \in \Gamma \} \). In order to make this set \( B \bowtie \Gamma \) a semigroup, we need to introduce a multiplication “\(*\)” defined on the set \( B \bowtie \Gamma \) by the following mapping. Firstly, we define a mapping \( \varphi : \Gamma \rightarrow \text{End}(B) \) by \( \gamma \mapsto \sigma_\gamma \), where \( \sigma_\gamma \in \text{End}(B) \) which is the endomorphism semigroup on \( B \). This mapping satisfies
following properties:

(P1) Absorbing: for each $\gamma \in \Gamma$ and $\alpha \in Y$, we have $B_\alpha \sigma_\gamma \subseteq B_{(\gamma \alpha)^!}$. In particular, if $\gamma \in Y$, then $\sigma_\gamma$ is an inner endomorphism on $B$ such that $\sigma_\gamma = fe$ for some $f \in B_\gamma$ and all $e \in B$.

(P2) Focusing: for $\alpha, \beta \in \Gamma$ and $f \in B_{(\alpha \beta)^!}$, we have $\sigma_\beta \sigma_\alpha \delta_f = \sigma_{\alpha \beta} \delta_f$, where $\delta_f$ is an inner endomorphism induced by $f$ on $B$ satisfying $h \delta_f = fh$ for all $h \in B$.

(P3) Homogenizing: for $e \in B_{\omega^!}, g \in B_{\tau^!}$ and $h \in B_{\xi^!}$, if $\omega \tau = \omega \xi$ and $eg\omega = eh\omega$, then $fg\omega = fh\omega$, for any $f \in B_{\omega^!}$.

(P4) Idempotent connecting: assume that for any $\omega \in \Gamma$, $\eta$ is the connecting isomorphism which maps $\langle \omega^! \rangle$ to $\langle \omega^* \rangle$ by $\alpha \mapsto \alpha \eta$. If $(e, \omega^!)$ and $(f, \omega^*) \in B \bowtie \Gamma$, then there is a bijection $\theta : \langle e \rangle \to \langle f \rangle$ such that

(i) $e \theta = f$ and $g = e (g \theta) \omega^*$, for $g \in \langle e \rangle$;

(ii) for $g \in \langle e \rangle$ and $\alpha \in \langle \omega^! \rangle, (g, \alpha) \in B \bowtie \Gamma$ if and only if $(g \theta, \alpha \eta) \in B \bowtie \Gamma$.

Equipped with the above mapping $\varphi$, we hence define a multiplication "*" on $B \bowtie \Gamma$ by

$$(e, \omega) * (f, \tau) = (ef\omega, \omega \tau)$$

for any $(e, \omega), (f, \tau) \in B \bowtie \Gamma$, where $f\omega = f\sigma_\omega$.

It can be verified that the multiplication "*" defined above for the set $B \bowtie \Gamma$ is associative. We call the semigroup a left wreath product of a left regular band $B$ and a type A semigroup $\Gamma$ under a mapping $\varphi$, denoted by $B \bowtie \varphi \Gamma$.

We are now going to establish a structure theorem for $L^*$-inverse semigroups.

**Theorem 4.1.** ([26], Theorem 4.1) A semigroup $S$ is an $L^*$-inverse semigroup if and only if $S$ is a left wreath product of a left regular band $B$ and a type A semigroup $\Gamma$.

In [27], we call an IC abundant semigroup $S$ a $Q^*$-inverse semigroup if the set of its idempotents $E$ forms a regular band, i.e. $E$ satisfies the identity $efege = efge$, for all $e, f$ and $g$ in $E$.

Suppose that $S$ is a $Q^*$-inverse semigroup whose set of idempotents $E$ forms a regular band. Denote the $J$-class containing the element $e \in E$ by $E(e)$. We first have the following result.

**Theorem 4.2.** ([27], Theorem 3.2) If an equivalence relation $\delta$ on $S$ is defined by $a \delta b$ if and only if $b = eaf$ and $a = gbh$ for some $e \in E(a^+), f \in E(a^*), g \in E(b^+)$ and $h \in E(b^*)$, then the equivalence relation $\delta$ is the smallest type A good congruence on $S$. 17
Let $S$ be a $Q^*$-inverse semigroup with a regular band of idempotents $E$. Define relations $\mu_l$ and $\mu_r$ on $S$ as follows:

$$(a, b) \in \mu_l \iff (xa, xb) \in L^* \quad (x \in E),$$

$$(a, b) \in \mu_r \iff (ax, bx) \in R^* \quad (x \in E).$$

Put $\rho_1 = \delta \cap \mu_r$ and $\rho_2 = \delta \cap \mu_l$ on $S$ (see [27]). We are now able to establish the following theorem for $Q^*$-inverse semigroups.

To obtain structure theory for $Q^*$-inverse semigroups, the concept of the wreath product of semigroups was introduced by Ren and Shum in [27] as follows:

In the wreath product of semigroups, we need the following ingredients:

(a) $Y$: a semilattice.

(b) $\Gamma$: a type $A$ semigroup whose set of idempotents is the semilattice $Y$.

(c) $I$: a left regular band such that $I = \bigcup_{\alpha \in Y} I_{\alpha}$, where $I_{\alpha}$ is a left zero band for all $\alpha \in Y$.

(d) $\Lambda$: a right regular band such that $\Lambda = \bigcup_{\alpha \in Y} \Lambda_{\alpha}$, where $\Lambda_{\alpha}$ is a right zero band for all $\alpha \in Y$.

We now form the following sets:

$I \Join \Gamma = \{(e, \omega) \mid \omega \in \Gamma, e \in I_{\omega^+}\}$,

$\Gamma \Join \Lambda = \{(\omega, i) \mid \omega \in \Gamma, i \in \Lambda_{\omega^*}\}$,

and

$I \Join \Gamma \Join \Lambda = \{(e, \omega, i) \mid \omega \in \Gamma, e \in I_{\omega^+} \text{ and } i \in \Lambda_{\omega^*}\}$.

Since $\omega \in \Gamma$ and $\Gamma$ is a type $A$ semigroup, there are some idempotents $\omega^+ \in R^*_\omega(\Gamma) \cap E(\Gamma)$ and $\omega^* \in L^*_\omega(\Gamma) \cap E(\Gamma)$. Also since the set of idempotents of $\Gamma$ forms a semilattice, $\omega^+$ and $\omega^*$ are in $Y$. This illustrates that the sets $I \Join \Gamma, \Gamma \Join \Lambda$, and $I \Join \Gamma \Join \Lambda$ are well-defined. We only need to define an associative multiplication on the set $I \Join \Gamma \Join \Lambda$ so that the set $I \Join \Gamma \Join \Lambda$ under the multiplication turns out to be a semigroup.

Before we define a multiplication on $I \Join \Gamma \Join \Lambda$, we need to give a description for the structure mappings.

Define a mapping $\varphi : \Gamma \to \text{End}(I)$ by $\gamma \mapsto \sigma_\gamma$ for $\gamma \in \Gamma$ and $\sigma_\gamma \in \text{End}(I)$ satisfying the following conditions:

(P1) For each $\gamma \in \Gamma$ and $\alpha \in Y$, we have $I_{\alpha} \sigma_\gamma \subseteq I_{(\gamma \alpha)^+}$. In particular, if $\gamma \in Y$ then $\sigma_\gamma$ is an inner endomorphism on $I$ such that there exists $g \in I_{\gamma}$ with $e^{\gamma \gamma} = ge$, for all
$e \in I$, where $e^{\sigma_\gamma}$ denotes $e\sigma_\gamma$.

(P2) For $\alpha, \beta \in \Gamma$ and $f \in I_{(\alpha\beta)}^\dagger$, we have $\sigma_\beta \sigma_\alpha \delta_f = \sigma_\alpha \delta_f \sigma_\beta$, where $\delta_f$ is an inner endomorphism induced by $f$ on $I$ satisfying $h^{\delta_f} = fh = fhf$, for all $h \in I$.

(P3) For $e \in I_{\omega_1}, g \in I_{\tau_1}$ and $h \in I_{\xi_1}$, if $\omega_\tau = \omega_\xi$ and $eg^{\omega_\tau} = eh^{\omega_\tau}$, then $fg^{\omega_\tau} = fh^{\omega_\tau}$, for all $f \in I_{\omega_\tau}$.

(P4) Assume that for any $\omega \in \Gamma, \eta$ is the connecting isomorphism which maps $\langle \omega^1 \rangle$ to $\langle \omega^* \rangle$ by $\alpha \mapsto \alpha \eta$. If $(e, \omega^1)$ and $(f, \omega^*) \in I \bowtie \Gamma$, then there is a bijection $\theta : \langle e \rangle \to \langle f \rangle$ such that

(i) $ef = f$ and $ge^{\omega_\tau} = e(g\theta)^{\omega_\tau}$ for any $g \in \langle e \rangle$ and $\alpha \in \langle \omega^1 \rangle$.

(ii) For any $g \in \langle e \rangle$ and $\alpha \in \langle \omega^1 \rangle$, $(g, \alpha) \in I \bowtie \Gamma$ if and only if $(g\theta, \alpha \eta) \in I \bowtie \Gamma$.

Similarly, define a mapping $\psi : \Gamma \to \text{End}(\Lambda)$ by $\gamma \mapsto \rho_\gamma$ for $\gamma \in \Gamma$ and $\rho_\gamma \in \text{End}(\Lambda)$ satisfying the following conditions:

(P1)' For each $\gamma \in \Gamma$ and $\alpha \in Y$, we have $\Lambda_\alpha \rho_\gamma \subseteq \Lambda_{(\alpha \tau)^\gamma}$. In particular, if $\gamma \in Y$, then $\rho_\gamma$ is an inner endomorphism on $\Lambda$ such that there exists $i \in \Lambda_\gamma$ with $ji^\rho_\gamma = ji$ for all $j \in \Lambda$, where $ji^\rho_\gamma$ denotes $j\rho_\gamma$.

(P2)' For $\alpha, \beta \in \Gamma$ and $i \in \Lambda_{(\alpha\beta)^*}$, we have $\rho_\alpha \rho_\beta \varepsilon_i = \rho_\alpha \varepsilon_i \varepsilon_i$, where $\varepsilon_i$ is an inner endomorphism induced by $i$ on $\Lambda$ such that $j^\varepsilon_i = ji = ji^\varepsilon_i$ for any $j \in \Lambda$.

(P3)' For $i \in \Lambda_\omega^*, j \in \Lambda_\tau^*$ and $k \in \Lambda_\xi^*$, if $\tau = \xi_\omega$ and $j^\rho_i = k^\rho_i$, then $j^\rho_\omega m = k^\rho_\omega m$ for all $m \in \Lambda_\omega^*$.

(P4)' Assume that for any $\omega \in \Gamma, \eta$ is the connecting isomorphism which maps $\langle \omega^1 \rangle$ to $\langle \omega^* \rangle$ by $\alpha \mapsto \alpha \eta$. If $(\omega^1, j)$ and $(\omega^*, i) \in I \bowtie \Lambda$, then there is a bijection $\theta' : \langle i \rangle \to \langle j \rangle$ such that the following conditions hold:

(i) $j\theta' = i, k^\rho_i = i^{\rho \eta}(k\theta')$, for any $k \in \langle j \rangle$ and $\alpha \in \langle \omega^1 \rangle$;

(ii) For any $k \in \langle i \rangle$ and $\alpha \in \langle \omega^1 \rangle$, $(\alpha, k) \in I \bowtie \Lambda$ if and only if $(\alpha \eta, k\theta') \in I \bowtie \Lambda$.

After gluing up the above components $I, \Gamma$ and $\Lambda$ together with the mappings $\varphi$ and $\psi$, we now define a multiplication on the set $I \bowtie \varphi \bowtie \psi \Lambda$ by

$$(e, \omega, i) \ast (f, \tau, j) = (ef^{\omega_\tau}, \omega_\tau, i^\rho_\tau),$$

for any $(e, \omega, i), (f, \tau, j) \in I \bowtie \Gamma \bowtie \Lambda$, where $f^{\omega_\tau} = f\sigma_\omega$ and $i^\rho_\tau = \rho_\tau$.

By using the properties (P1), (P2), (P1)', and (P2)', we can easily verify that the above multiplication "\ast" on $I \bowtie \varphi \bowtie \psi \Lambda$ is associative. We now call the above constructed semigroup the wreath product of $I, \Gamma$ and $\Lambda$ with respect to $\varphi$ and $\psi$, and denote it by $Q = I \bowtie \varphi \bowtie \psi \Lambda$.

**Theorem 4.3.** ([27], Theorem 4.4) The wreath product $I \bowtie \varphi \bowtie \psi \Lambda$ of a left regular band $I$, a type $A$ semigroup $\Gamma$ and a right regular band $\Lambda$ with respect to the mappings $\varphi$ and $\psi$ is a $Q^*$-inverse semigroup.
Conversely, every $Q^*$-inverse semigroup $S$ can be expressed by a wreath product of $I \bowtie \varphi \Gamma \bowtie \psi \Lambda$.

**Remark 1.** The class of $Q^*$-inverse semigroups contains several interesting classes of semigroups as its special subclasses. We only discuss some of these special subclasses as follows.

(a) $L^*$-inverse semigroups and $R^*$-inverse semigroups

By Theorem 4.3, a $Q^*$-inverse semigroup $S$ can be expressed as a wreath product $I \bowtie \varphi \Gamma \bowtie \psi \Lambda$ of $I$, $\Gamma$, and $\Lambda$ with respect to the mappings $\varphi$ and $\psi$, where $\Gamma$ is a type $A$ semigroup, $I$ and $\Lambda$ are respectively a left regular band and a right regular band. In Theorem 4.3, if $\Lambda = \emptyset$, then $S = I \bowtie \Gamma$, which is an $L^*$-inverse semigroup. Similarly, if we let $I = \emptyset$, then $\Gamma \bowtie \psi \Lambda$ becomes an $R^*$-inverse semigroup. Thus, the class of $L^*$-inverse semigroups and the class of $R^*$-inverse semigroups are two special subclasses of the class of $Q^*$-inverse semigroups. In this case, we can easily reobtain Theorem 4.2 for structure of $L^*$-inverse semigroups, as a corollary of Theorem 4.3.

(b) Quasi-inverse semigroups

We know that a quasi-inverse semigroup is a regular semigroup whose set of idempotents forms a regular band. It is clear that a quasi-inverse semigroup is a special $Q^*$-inverse semigroup.

When $S$ is a quasi-inverse semigroup, we can define a relation $\delta$ on $S$ by $a \delta b$ if and only if $b = eaf$ for some $e \in E(aa')$ and $f \in E(a'a)$, where $a'$ is an inverse element of $a$. It can be immediately seen from [26] that $\delta$ is the smallest inverse semigroup congruence on $S$ and so $\Gamma = S/\delta$ is the greatest inverse semigroup homomorphism image of $S$. Obviously, the inverse semigroup $\Gamma = S/\delta$ must be a type $A$ semigroup whose set of idempotents forms a semilattice. As a result, a wreath product $I \bowtie \varphi \Gamma \bowtie \psi \Lambda$ of $S$, regarded as a $Q^*$-inverse semigroup, can be simplified by using the so called **half-direct product** (in brevity, $H.D.$-product) of a quasi-inverse semigroup given by M. Yamada in [38] as described in the following Theorem 4.4.

**Theorem 4.4.** ([38], Theorem 6) Let $S$ be a quasi-inverse semigroup whose set of idempotents forms a regular band $E$. Let $\delta$ be the smallest inverse congruence on $S$ such that $\Gamma = S/\delta$ is the greatest inverse semigroup induced by $\delta$ and let $Y$ be the semilattice of $\Gamma$. Define the congruences $\eta_1, \eta_2$ on $E$ by $e \eta_1 f$ if and only if $eRf$; $e \eta_2 f$ if and only if $eLf$, respectively.
For $X \subseteq E$, write $\tilde{X} = \{ \tilde{e} \mid e \in X \}$ and $\hat{X} = \{ \hat{e} \mid e \in X \}$, where $\tilde{e}$ and $\hat{e}$ are the $\eta_1$-class and the $\eta_2$-class containing $e \in X$, respectively. Then the following statements hold:

(i) $E/\eta_1 = \tilde{E}$ is a left regular band such that $\tilde{E} = \bigcup_{a \in Y} \tilde{E}_a$, where every $\tilde{E}_a$ is a left zero band; $E/\eta_2 = \hat{E}$ is a right regular band such that $\hat{E} = \bigcup_{a \in Y} \hat{E}_a$, where each $\hat{E}_a$ is a right zero band, for every $a \in Y$.

(ii) $S$ is isomorphic to an H.D.-product of $\tilde{E}, \Gamma$ and $\hat{E}$ with respect to the mappings $\varphi'$ and $\psi'$, respectively. Conversely, any H.D.-product of a left regular band $I = \bigcup_{a \in Y} \Lambda_a$, an inverse semigroup $\Gamma$ and a right regular band $\Lambda = \bigcup_{a \in Y} \Lambda_a$ with respect to the mappings $\varphi'$ and $\psi'$ is a quasi-inverse semigroup $S$, where $\Gamma$ is the greatest inverse semigroup homomorphic image of $S$ and $Y$ is the semilattice of idempotents of $\Gamma$.

K. Kimura in 1958 first considered [15] the spined product of semigroups as follows: if $T_1$ and $T_2$ are two semigroups having a common homomorphic image $H$, and if $\phi : T_1 \to H$ and $\psi : T_2 \to H$ are homomorphisms onto $H$, then the spined product of $T_1$ and $T_2$ with respect to $H, \phi$ and $\psi$ is defined by the set $\{(t_1, t_2) \in T_1 \times T_2 \mid t_1\phi = t_2\psi\}$. In particular, we denote the spined product of $T_1$ and $T_2$ with respect to $H, \phi$ and $\psi$ by $T_1 \times_{\phi, H} T_2$.

For the $Q^*$-inverse semigroups, we have the following constructions.

**Theorem 4.5** ( [27], Theorem 5.1) A semigroup $S$ is a $Q^*$-inverse semigroup if and only if $S$ is a spined product of an $L^*$-inverse semigroup $S_1 = I \bowtie_{\varphi} \Gamma$ and an $R^*$-inverse semigroup $S_2 = \Gamma \bowtie_{\psi} \Lambda$ with respect to a type $A$ semigroup $\Gamma$.

The relationship between the $L^*$-inverse semigroup $S_1$, the $R^*$-inverse semigroup $S_2$ and the type $A$ semigroup $\Gamma$ can be expressed as the following picture, where

$$S = S_1 \times_{\Theta} S_2 = \{((e, \omega), (\omega, i)) \in S_1 \times S_2 \mid (e, \omega)\Theta_1 = (\omega, i)\Theta_2\}$$

with a multiplication “$\circ$” given by

$$((e, \omega), (\omega, i)) \circ ((f, \tau), (\tau, j)) = ((ef^{\omega\omega}, \omega\tau), (\omega\tau, i^{\omega\omega}j)).$$
References


[38] Yamada, M., Orthodox semigroups whose idempotents satisfy a certain identity, Semigroup Forum, 6 (1973), 113–128.
